Equations of first order and first degree

Separation of Variables

A special case of an exact equation occurs when, in the equation $Mdx + Ndy = 0$, M is a function of x alone and N is a function of y alone.

The equation $f_1(x)\,dx + f_2(y)\,dy = 0$ is said to have variable seperable. It's general solution is

$$\int f_1(x)\,dx + \int f_2(y)\,dy = c$$

Example ① Solve $\dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dx}{\sqrt{1-x^2}} = 0$

Solution:→

$$\therefore \int \frac{dy}{\sqrt{1-y^2}} + \int \frac{dx}{\sqrt{1-x^2}}\,dx = 0$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = c$$

$$\Rightarrow \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\} = c$$

$$\Rightarrow x\sqrt{1-y^2} + y\sqrt{1-x^2} = \log c = k$$

Where k is the arbitrary constant.

Example ② Solve $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

Solution:→

$$\frac{x\,dx}{\sqrt{1-x^2}} + \frac{y\,dy}{\sqrt{1-y^2}} = 0$$

Integrating, we have.

$$\cancel{\int\frac{x\,dx}{\sqrt{1-x^2}}+\int\frac{y\,dy}{\sqrt{1-y^2}}=0}$$

$$\int\frac{x\,dx}{\sqrt{1-x^2}}+\int\frac{y\,dy}{\sqrt{1-y^2}}=0$$

Let $1-x^2=t^2$, $1-y^2=z^2$

$\Rightarrow -2x\,dx=2t\,dt$ & $-2y\,dy=2z\,dz$

$$\therefore \int\frac{-t\,dt}{\sqrt{t^2}}+\int\frac{-z\,dz}{\sqrt{z^2}}=c$$

$$\Rightarrow \int dt+\int dz=-c=k$$

$$\Rightarrow t+z=k$$

$$\therefore \sqrt{1-x^2}+\sqrt{1-y^2}=k$$

Ans.

Example (3) $(x^2+1)(y^2-1)\,dx+xy\,dy=0$

Soln:-

$$\because (x^2+1)(y^2-1)\,dx+xy\,dy=0$$

$$\Rightarrow \frac{x^2+1}{x}\,dx+\frac{y}{y^2-1}\,dy=0$$

Integrating, we have

$$\int\left(x+\frac{1}{x}\right)dx+\frac{1}{2}\int\frac{2y}{y^2-1}\,dy=0$$

$$\Rightarrow \frac{x^2}{2}+\log x+\frac{1}{2}\log(y^2-1)=c$$

Where c is constant.

$\Rightarrow x^2 + 2\log x + \log(y^2-1) = 2c$

$\Rightarrow x^2 + \log x^2 + \log(y^2-1) = \log k$ (say)

$\therefore x^2 = \log \frac{k}{x^2(y^2-1)}$

or, $e^{x^2} = \frac{k}{x^2(y^2-1)}$

$\Rightarrow y^2 - 1 = \frac{k}{x^2} \cdot e^{-x^2}$

$\Rightarrow y^2 = 1 + \frac{k}{x^2} e^{-x^2}$ Ans.

Example ④ solve $\frac{dy}{dx} = \sin(x+y)$ —①

Solution:→ Let $x + y = v$

$\Rightarrow \frac{dy}{dx} + 1 = \frac{dv}{dx}$

$\therefore$ From ①, we have.

$\frac{dv}{dx} - 1 = \sin v$

$\Rightarrow \frac{dv}{dx} = 1 + \sin v$

$\Rightarrow \frac{dv}{1+\sin v} = dx$

integrating, we have

$\int \frac{dv}{1+\sin v} = \int dx$

$$\Rightarrow x + c = \int \frac{dv}{1+\sin v}$$

$$= \int \frac{1 - \sin v}{\cos^2 v}\, dv$$

$$= \int \left(\frac{1}{\cos^2 v} - \frac{1}{\cos v} \cdot \frac{\sin v}{\cos v} \right) dv$$

$$= \int (\sec^2 v - \sec v \cdot \tan v)\, dv$$

$$= \tan v - \sec v$$

$\therefore$ The general solution is,

$$\tan(x+y) - \sec(x+y) = x + c$$

Ans.

Example ⑤: Solve

$$(1-x)\,dy - (1+y)\,dx = 0$$

Example ⑥ :- Solve

$$xy^2\,dy - y^3\,dx + y^2\,dy = dx$$

Example ⑦ Solve

$$\frac{dy}{dx} + 2xy = x^2 + y^2$$